The Bekenstein-Hawking Entropy of Higher-Dimensional Rotating Black Holes

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Abstract

A black hole can be regarded as a thermodynamic system described by a grand canonical ensemble. In this paper, we study the Bekenstein-Hawking entropy of higher-dimensional rotating black holes using the Euclidean path-integral method of Gibbons and Hawking. We give a general proof demonstrating that ignoring quantum corrections, the Bekenstein-Hawking entropy is equal to one-fourth of its horizon area for general higher-dimensional rotating black holes.

1 Introduction

The Bekenstein-Hawking entropy of a black hole, not considering quantum corrections, is equal to one-fourth of its horizon area.[1),2)] This conclusion can be obtained using many different approaches. A classical approach to derive the entropy of a black hole was formulated by Gibbons and Hawking in terms of the Euclidean path-integral and grand canonical ensemble methods.[3),4)] In this paper, we study the entropy of higher-dimensional rotating black holes following the method of Gibbons and Hawking. To make this work self-contained, we first review the method of Gibbons and Hawking for determining the entropy of a black hole.[3),4)]

A black hole can be regarded as a thermodynamic system described by a grand canonical ensemble, because it possesses an explicit temperature and is in a state of thermal equilibrium with respect to radiation. It exchanges particles and energy with the surrounding spacetime. We can write down its grand partition function Z, thermodynamic potential W and entropy S:

$$Z = \exp[-W] = \mathrm{Tr}\exp[-(\beta\hat{H} - \mu_i\hat{C}_i)] \ , \tag{1}$$

$$W = E - TS - \mu_i C_i \ , \tag{2}$$

$$S = \beta(E - \mu_i C_i) + \ln Z \ . \tag{3}$$

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Here $\frac{1}{\beta}$ is the temperature and μ_i represents the chemical potentials. The partition function (1) can also be written in the Euclidean path integral form. This is given by

$$Z = \int D[g,\phi] \exp(-I_E[g,\phi]) \ , \tag{4}$$

where ϕ represents matter fields, including the electromagnetic fields of charged black holes. A Wick rotation has been performed in order to realize the Euclidean form in (4). We can compute Z perturbatively, and to first order, we obtain

$$Z = \exp[-I_E^{\infty}] \ , \tag{5}$$

where

$$I_E^{\infty} = \frac{1}{16\pi}\int_M [(R-2\Lambda)+\mathcal{L}_{\text{matter}})] + \frac{1}{8\pi}\int_{\partial M}[K] \tag{6}$$

is the on-shell Euclidean action, and $[K] = K - K_0$ is the difference between the extrinsic curvature of the spacetime manifold and that of a reference background spacetime. The upper index ∞ of (6) means that the boundary of the spacetime manifold lies only at $r=\infty$, i.e., in (6) ∂M lies only at $r=\infty$. The reason for this is that we are considering the black hole as a thermal equilibrium system. Thus it is reasonable that we take the metric of the spacetime manifold in the form of the largest extension, as in the case of the Kruskal metric of a Schwarzschild black hole, because the singularity on the horizon is only apparent and can be moved away.[5),6)] Also I_E is expected to be invariant under a general coordinate transformation. The above arguements also hold for the metric of a higher-dimensional rotating black hole. For the metric of a higher-dimensional rotating black hole, the above consideration is also tenable.

Next, we must treat the term $\beta(E-\mu_i C_i)$ in (3). For this purpose, we consider the quantum transition amplitude between two space-like hyperspaces in a black hole's spacetime manifold in the Euclidean time formalism. This is given by

$$\langle\tau_1|\tau_2\rangle = \langle\tau_1|e^{-(\tau_2-\tau_1)\hat{H}}|\tau_1\rangle \ . \tag{7}$$

Under the condition that the fluctuations of energy are relatively small, i.e., $\frac{\langle E\rangle^2-\langle E^2\rangle}{\langle E\rangle^2} \ll 1$, we can expand (7) while ignoring the fluctuations. This yields

$$\langle\tau_1|\tau_2\rangle = e^{-(\tau_2-\tau_1)E} \ . \tag{8}$$

For example, for a Schwarzschild black hole, we have $\langle E\rangle^2 = M^2$, $\langle E\rangle^2 - \langle E^2\rangle = \frac{M_P^2}{8\pi}$, and have the above condition is satisfied if $M \gg M_P$. As pointed out by Kallosh et al.,[7)] (8) can be generalized to the case in which there exist multiple conserved charges C_i. To introduce Lagrange multipliers μ_i and to consider constrained imaginary time evolution so that only metrics with designated charges are considered in the path integral, we obtain

$$\langle\tau_1|\tau_2\rangle = e^{-(\tau_2-\tau_1)(E-\mu_i C_i)} \ . \tag{9}$$

On the other hand, the quantum transition amplitude $\langle\tau_1|\tau_2\rangle$ can be obtained using the Feynman path integral formulation as

$$\langle\tau_1|\tau_2\rangle = \int \ D[g,\phi]e^{-I[g,\phi]} \ . \tag{10}$$

To first-order, this gives

$$\langle \tau_1 | \tau_2 \rangle = e^{-I^{\infty}_{E,h}} \ , \tag{11}$$

where again $I^{\infty}_{E,h}$ is given by (6). Here the subscript h on the Euclidean action means that the black hole's horizon is considered as another spacetime boundary. We include this boundary because we study the quantum transition amplitude in the black hole's spacetime manifold. No physical information can escape from the horizon of a black hole. Therefore we need to take the horizon as another spacetime boundary. Thus the black hole's horizon also contributes to the Euclidean action integral of (6) in (11). Comparing (9) and (11), and fixing

$$\tau_2 - \tau_1 = \beta \ , \tag{12}$$

we obtain

$$\beta(E - \mu_i C_i) = I^{\infty}_{E,h} \ . \tag{13}$$

This relation results from the fact that β is the period of the imaginary time for the black hole's spacetime. Inserting (5), (6) and (13) into (3), we obtain

$$S = \frac{1}{8\pi}\left(\int_h^{\infty}[K] - \int^{\infty}[K]\right) = -\frac{1}{8\pi}\int_h [K] \ . \tag{14}$$

Thus we arrive at the conclusion of Gibbons and Hawking: A black hole's entropy, not considering quantum corrections, is determined by the gravitational surface term. This conclusion is valid for spherically symmetric black holes, as well as for charged and rotating black holes. The differences among the entropies of different types of black holes result only from their different conserved charges C_i, because the matter fields parts are canceled in the above derivation.

2 The Bekenstein-Hawking entropy of higher-dimensional rotating black holes

Using certain surface terms of the gravitational action, one can derive the relation $S = \frac{1}{4}A$ for a black hole's entropy. The derivation of this relation is given in Refs. 3), 7) and 8) for certain spherically symmetric black holes. We seek to derive the relation $S = \frac{1}{4}A$ for higher-dimensional rotating black holes in this paper. For this purpose, we need to derive the explicit form of the gravitational surface term K.

There are many definitions of the gravitational surface term in the literature (e.g., see Refs. 3), 4) and 7)-10)). Usually, the surface term K can be defined as the trace of the second fundamental form on the horizon. However, different forms of the gravitational surface terms are equivalent for the purpose of determining a black hole's entropy according to the formula (14). This is because the gravitational surface terms are total derivatives decomposed from the Einstein gravitational action. For different choices of the surface term, their differences are also total derivatives. Clearly, the field equations are not changed when a total derivative is added to the gravitational action. Therefore, the metric of a black hole is not changed when a total derivative is added to the gravitational action. Hence, different choices of the

surface term yield identical forms of the black hole entropy formula (14). We find that the surface term of Landau and Lifshitz[10] is most convenient for deriving the relation $S = \frac{1}{4}A$ for higher-dimensional rotating black holes, as in the case of spherically symmetric black holes.[7),8)]

According to Landau and Lifshitz,[10] the gravitational action can be decomposed in two parts

$$\sqrt{-g}R = \sqrt{-g}G + \partial_\mu(\sqrt{-g}\omega^\mu) \ , \tag{15}$$

where the first term contains no second derivatives. Omitting the total derivative $\partial_\mu(\sqrt{-g}\partial_\nu g^{\mu\nu})$ in the action, ω^μ is obtained as

$$\omega^\mu = -\frac{2}{\sqrt{-g}}\left(\frac{\partial\sqrt{-g}}{\partial x^\nu}\right) g^{\mu\nu} - \frac{\partial g^{\nu\mu}}{\partial x^\nu} \ . \tag{16}$$

Therefore, the surface term is given by

$$K = \frac{1}{2}\omega^\mu n_\mu \ , \tag{17}$$

where n_μ is the space-like outward-pointing unit normal vector on the horizon. Usually, in the case of a charged black hole, there also exist certain surface terms that are related to the matter fields[11),12)] as

$$K_{\text{matter}} = \frac{1}{4\pi}\int_h d^{D-1}x\sqrt{g_{D-1}}n_\mu F^{\mu\nu}A_\nu \ , \tag{18}$$

where D is the dimension of the spacetime manifold. However, in the derivation of (14), the contributions from the matter fields are canceled. Therefore we need not to consider the surface terms that related to different matter fields in the entropy formula (14). However, these terms may have an effect in the analysis of quantum corrections to the black hole's entropy.

Now we rewrite (14) in the explicit form

$$S = -\frac{1}{8\pi}\int_h d^{D-1}x\sqrt{g_{D-1}}(K - K_0) \ , \tag{19}$$

where D is the dimension of the spacetime manifold, g_{D-1} is the determinant of the metric on the horizon, and K_0 comes from the background Minkowski spacetime in which the black hole is embedded. The general solution for a higher-dimensional rotating black hole has been obtained by Myers and Perry.[13] Here we adopt the parameterized form derived by Cvetič and Youm[14] for the purpose of deriving the relation $S = \frac{1}{4}A$. With this form, the metric for a higher-dimensional rotating black hole is given by

$$\begin{aligned} ds^2 \ &= \ g_{\tau\tau}d\tau^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\psi_i\psi_i}d\psi_i^2 + 2g_{\theta\psi_i}d\theta d\psi_i + \sum_{i<j} 2g_{\psi_i\psi_j}d\psi_i d\psi_j \\ &\quad + g_{\phi_i\phi_i}d\phi_i^2 + \sum_{i<j} 2g_{\phi_i\phi_j}d\phi_i d\phi_j + 2g_{\tau\phi_i}d\tau d\phi_i \ . \end{aligned} \tag{20}$$

Here we have written the metric in Euclidean form, because formula (14) is derived for the Euclidean form of the black hole metric. The Euclidean form of the metric can be obtained

from the corresponding Lorentzian metric by rotating the time axis and the parameters.[15),16)] In (20), i and j on ϕ run from 1 to $[\frac{D-1}{2}]$. For the even-dimensional case, i and j on ψ run from 1 to $[\frac{D-3}{2}]$. For the odd-dimensional case, i and j on ψ run from 1 to $[\frac{D-5}{2}]$. Obviously, the black hole rotates in each of the ϕ_i directions, and there are $[\frac{D-1}{2}]$ angular momentum components. The area of the horizon is given by the integral

$$A = \int d\theta \prod d\psi_i \prod d\phi_j \left[\sqrt{\det g_{ab}}\right]\Big|_{r=r_h} . \tag{21}$$

Here we use r_h to denote the radius of the horizon. Also, we use g_{ab} to denote the metrics of the $d\theta^2$, $d\psi_i^2$, $d\theta d\psi_i$, $d\psi_i d\psi_j$, $d\phi_i^2$ and $d\phi_i d\phi_j$ terms of the metric given in (20). We regard the horizon to be the outer horizon of a rotating black hole. In addition, we consider non-extremal black holes in this paper. The entropy of an extremal black hole is zero, according to Refs. 8), 17) and 18). This can also be seen from (19). Because the radius r_h of the horizon lies at the space-like infinity for an extremal black hole, we obtain $K - K_0 = 0$ in (19).

We denote the angular velocity of the horizon corresponding to the coordinate ϕ_i as Ω_i. In order to simplify the derivation, we carry out the coordinate transformation

$$\phi_i \to \phi_i' = \phi_i - \Omega_i \tau , \tag{22}$$

which means that the observer is co-rotates with the outer horizon. The metric (20) now becomes

$$\begin{aligned} ds^2 &= G_{\tau\tau} d\tau^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\psi_i\psi_i} d\psi_i^2 + 2g_{\theta\psi_i} d\theta d\psi_i + \sum_{i<j} 2g_{\psi_i\psi_j} d\psi_i d\psi_j \\ &\quad + g_{\phi_i\phi_i} d\phi_i'^{\,2} + \sum_{i<j} 2g_{\phi_i\phi_j} d\phi_i' d\phi_j' + 2g_{\tau\phi_i'} d\tau d\phi_i' , \end{aligned} \tag{23}$$

where

$$\begin{aligned} G_{\tau\tau} &= g_{\tau\tau} + g_{\phi_i\phi_i}\Omega_i^2 + \sum_{i<j} 2g_{\phi_i\phi_j}\Omega_i\Omega_j + 2g_{\tau\phi_i}\Omega_i , \\ g_{\tau\phi_i'} &= \Omega_j g_{\phi_i\phi_j} + g_{\tau\phi_i} . \end{aligned} \tag{24}$$

In the reference system co-rotating with the horizon, the metric on the horizon is static. Therefore, $g_{\tau\phi_i'}$ is zero on the horizon, and the angular velocities at the horizon satisfy

$$g_{\tau\phi_i'}|_{r=r_h} = 0 , \tag{25}$$

under the condition $\det g_{\phi_i\phi_j} \neq 0$. Then according to $(\mathrm{A}\cdot 5)$, we have

$$G_{\tau\tau}|_{r=r_h} = 0 . \tag{26}$$

The metrics g_{rr}, $g_{\theta\theta}$, $g_{\psi_i\psi_i}$, $g_{\theta\psi_i}$, $g_{\psi_i\psi_j}$, $g_{\phi_i\phi_i}$ and $g_{\phi_i\phi_j}$ are unchanged under the coordinate transformation (22). For the metric (20), the quantity $g^{rr} = 1/g_{rr}$ is also unchanged. The horizons of the metrics (20) and (23) are determined by the relation $g^{rr} = 0$. Therefore, the coordinate transformation (22) does not change the location of the horizon.

Because $g_{\tau\phi_i'}$ is zero on the horizon, the determinant of the metric (23) on the horizon is

$$g_{D-1} = G_{\tau\tau} \det g_{ab} , \tag{27}$$

where g_{ab} denotes the metrics of the $d\theta^2$, $d\psi_i^2$, $d\theta d\psi_i$, $d\psi_i d\psi_j$, $d\phi_i'^{\,2}$ and $d\phi_i' d\phi_j'$ terms in the metric (23). From (19), the entropy of the metric (23) is given by

$$S = -\frac{1}{8\pi}\int_0^\beta d\tau \int d\theta \prod d\psi_i \prod d\phi_j' \; \frac{1}{2}[\omega^\mu n_\mu \sqrt{g_{D-1}}]|_{r=r_h} \ , \tag{28}$$

where $\beta = 2\pi/\kappa$ is the inverse Hawking temperature, and κ is the surface gravity. For the rotating black hole metrics (20) and (23), an expression for κ is given in Appendix A. We can see that for the metric (20), (19) is invariant with respect to the rotating coordinate transformation (22). [1] Therefore the entropy calculated with the metric (23) is equal to that calculated with the metric (20). Hence we conclude that the entropy of a rotating black hole is unchanged under the rotating coordinate transformation (22) of its metric.

The space-like outward-pointing normal vector on the horizon can be written $(0, \sqrt{g_{rr}}, 0, ..., 0)$. Hence, in (28), for ω^μ, we only need to obtain ω^1. We use 0, 1, ..., $D-1$ to represent τ, r, ..., ϕ_j. For the metric (23), the only non-zero component of $g^{1\nu}$ is g^{11}. Thus, from (16) we have

$$\omega^1 = -\frac{2}{\sqrt{g}}\left(\frac{\partial\sqrt{g}}{\partial r}\right) g^{rr} - \frac{\partial g^{rr}}{\partial r} \ . \tag{29}$$

The determinant of the metric (23) is given by

$$g = G_{\tau\tau} g_{rr} \det g_{ab} - g^2_{\tau\phi_i'}(...)_i \ , \tag{30}$$

where $(...)_i$ represents terms that we need not consider explicitly here. The reason that these need not be included here is that in (29), the partial derivative of $\sqrt{g}$ is with respect to r. In (30), g has two parts. We can expand $\sqrt{g}$ as

$$\sqrt{g} = \sqrt{G_{\tau\tau} g_{rr} \det g_{ab}}\left[1 - \frac{1}{2}\frac{g^2_{\tau\phi_i'}(...)_i}{G_{\tau\tau} g_{rr} \det g_{ab}} + ...\right] \ . \tag{31}$$

The second part of (30) contributes zero near the horizon, because the metrics $g_{\tau\phi_i'}$ are squares and the metrics $g_{\tau\phi_i'}$ are zero on the horizon, while we evaluate (28) on the horizon, we consider only the contribution from the horizon in (28). Therefore, in the following, for convenience, we only consider the first term in (30). This means that we consider the limit $r \to r_h$ in the integral of (28).

Dropping the second term in (30), we obtain

$$[\omega^\mu n_\mu] = -\frac{2}{\sqrt{g_{rr}}}\frac{\partial \ln\sqrt{G_{\tau\tau}\det g_{ab}}}{\partial r} - \frac{2(D-2)}{r} \ , \tag{32}$$

where the second term comes from the background flat spacetime. Thus from (28), we obtain

$$S = -\frac{1}{4\kappa}\int d\theta \prod d\psi_i \prod d\phi_j' \frac{1}{2}\left[\sqrt{g_{D-1}}\left(-\frac{2}{\sqrt{g_{rr}}}\frac{\partial \ln\sqrt{g_{D-1}}}{\partial r} - \frac{2(D-2)}{r}\right)\right]\Bigg|_{r=r_h} \ , \tag{33}$$

[1] This is because $d^{D-1}x\sqrt{g_{D-1}}$ is invariant for the metric (20) under the rotating coordinate transformation (22). This can be seen to compare the metric (23) with the metric (20). And $K - K_0$ is a scalar in (19).

where g_{D-1} is given by (27). The flat spacetime background term in (33) contributes zero, because $G_{\tau\tau} = 0$ on the horizon. The first term inside the integral in (33) is a limit of $0 \cdot \infty$. Because κ is a constant on the horizon,[19] we can move it inside the integral. Then, using the expression for κ given in (A $\cdot$ 7), we obtain

$$S = \frac{1}{4} \int d\theta \prod d\psi_i \prod d\phi'_j \left[\frac{1}{\sqrt{g_{rr}}} \frac{\partial \sqrt{G_{\tau\tau} \det g_{ab}}}{\partial r} \bigg|_{r=r_h} \cdot \frac{\sqrt{g_{rr}}}{\partial_r \sqrt{G_{\tau\tau}}} \bigg|_{r=r_h} \right] . \tag{34}$$

Because $\sqrt{G_{\tau\tau}}$ is zero on the horizon and $1/\kappa$ is finite, we finally obtain

$$S = \frac{1}{4} \int d\theta \prod d\psi_i \prod d\phi'_j \left[\sqrt{\det g_{ab}} \right] \Big|_{r=r_h} . \tag{35}$$

As stated above, g_{ab} denotes the metrics of the $d\theta^2$, $d\psi_i^2$, $d\theta d\psi_i$, $d\psi_i d\psi_j$, $d\phi_i'^{\,2}$ and $d\phi'_i d\phi'_j$ terms in the metric (23). Hence it is seen that the above integral is just the area of the horizon with the metric (23). Then note that g_{ab} here is the same as that in the metric (20), despite the coordinate transformation (22), i.e., the metrics (20) and (23) do not depend on ϕ_i and ϕ'_i. Then the above area of the horizon with the metric (23) is equal to that with the metric (20), which is given by (21). This means that the area of the horizon of a rotating black hole does not change under the rotating coordinate transformation (22). Thus, the Bekenstein-Hawking entropy of a rotating black hole with the metric (20) is given by

$$S = \frac{1}{4} \int d\theta \prod d\psi_i \prod d\phi_j \left[\sqrt{\det g_{ab}} \right] \Big|_{r=r_h} , \tag{36}$$

which is equal to one-fourth of the area of its horizon according to (21). This completes the proof of the relation $S = \frac{1}{4} A$ for higher-dimensional rotating black holes.

3 Conclusion

In this paper, we have studied the Bekenstein-Hawking entropy of higher-dimensional rotating black holes using the Euclidean path-integral method of Gibbons and Hawking. A black hole can be regarded as a thermodynamic system described by a grand canonical ensemble. With the Euclidean path-integral, grand canonical ensemble approach,[3),4)] Gibbons and Hawking found that the entropy of a black hole, not considering quantum corrections, is determined by the gravitational surface term. Using the gravitational surface term of Landau and Lifshitz,[10)] we gave a general proof demonstrating that the Bekenstein-Hawking entropy, not considering quantum corrections, is equal to one-fourth of the area of its horizon for general higher-dimensional rotating black holes. The explicit form of the area of the horizon for higher-dimensional rotating black holes in terms of the metrics of Myers and Perry[13)] was recently calculated by Jung et al.[20)]

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Appendix A

—— Surface Gravity for Higher-Dimensional Rotating Black Holes ——

For the metric (20), there exists the Killing field

$$\xi^\mu = \frac{\partial}{\partial \tau} + \sum \Omega_i \frac{\partial}{\partial \phi_i} \ , \tag{A $\cdot$ 1}$$

where the quantities Ω_i are the angular velocities of the horizon corresponding to the coordinates ϕ_i. Because ξ^μ is normal to the horizon, we have

$$\xi^\mu \xi_\mu = 0 \tag{A $\cdot$ 2}$$

on the horizon. For the metric (20), we obtain

$$\xi^\mu \xi_\mu = g_{\tau\tau} + g_{\phi_i\phi_i}\Omega_i^2 + \sum_{i<j} 2g_{\phi_i\phi_j}\Omega_i\Omega_j + 2g_{\tau\phi_i}\Omega_i \ . \tag{A $\cdot$ 3}$$

Next, we define

$$G_{\tau\tau} = g_{\tau\tau} + g_{\phi_i\phi_i}\Omega_i^2 + \sum_{i<j} 2g_{\phi_i\phi_j}\Omega_i\Omega_j + 2g_{\tau\phi_i}\Omega_i \ . \tag{A $\cdot$ 4}$$

Therefore, we have

$$G_{\tau\tau}\big|_{r=r_h} = 0 \ . \tag{A $\cdot$ 5}$$

The surface gravity $\kappa(r_h)$ is constant on the horizon.[19] Writing $\xi^\mu \xi_\mu = -\lambda^2$, the surface gravity can be determined from the equation[21]

$$\nabla^\mu(\lambda^2)\nabla_\mu(\lambda^2) = -4\kappa^2\lambda^2 \ . \tag{A $\cdot$ 6}$$

Thus, for the metric (20), we obtain

$$\kappa(r_h) = \lim_{r\to r_h} \frac{\partial_r \sqrt{G_{\tau\tau}}}{\sqrt{g_{rr}}} \ . \tag{A $\cdot$ 7}$$

The partial derivative here is taken before the limit, because $G_{\tau\tau}$ is zero on the horizon. From (A $\cdot$ 6), we see that κ is a scalar. Hence, the surface gravity of a rotating black hole is invariant under general coordinate transformations.

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